Security Issuance with Undersubscription Risk

David M. Frankel (Melbourne Business School)*

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Abstract

A successful security offering raises more revenue for investments that will be used to pay distributions on the securities. This can give rise to a coordination problem among investors, which the issuer can overcome by underpricing the security. In this setting, debt is optimal because it minimizes underpricing. In special cases, however, equity underpricing can be eliminated by share rationing or a minimum sales requirement (MSR). In general, the optimal MSR is decreasing in due diligence costs and in the availability of alternative funding sources.

JEL: G12, G14, G32.

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*200 Leicester Street, Carlton, VIC 3053, Australia. Email: d.frankel@mbs.edu. I thank seminar participants at Berkeley, Hebrew U., Melbourne, Monash, Northwestern, Paris School of Economics, Penn, Princeton, Sorbonne, Stanford GSB, Tel Aviv, UCL, and Wisconsin. I am also grateful for the helpful comments of Philip Bond, Harry DeAngelo, Peter DeMarzo, Darrell Duffie, Sivan Frenkel, Neal Galpin, Ilan Kremer, Nisan Langberg, Deborah J. Lucas, Gideon Saar, Keke Song, Eyal Winter, and Ming Yang.
1 Introduction

“Sometimes [an issuer in a public offering] ... is a comparatively new company, is making the ... offering to raise the capital necessary to begin or expand its activities, and the failure to receive it will substantially impair its ability to continue in business or to conduct necessary operations.” (U.S. Security and Exchange Commission [85])

Firms raise substantial capital for investments from issuing securities. A successful offering will thus let an issuer invest more, leading to a higher future cash flow from which to pay distributions on the securities. This can give rise to a coordination problem among investors, in which it is worthwhile to subscribe only if enough others are expected to do so. We present a new theory of underpricing and security design based on this idea.

Our first result is that standard debt is optimal as it is the least underpriced security. Pecking order behavior à la Myers [67] thus results from an issuer’s desire to mitigate underpricing. This is consistent with the empirical evidence that debt is more common and less underpriced than equity. We next study equity offerings. These are often rationed, which leads to oversubscription. They may also have a minimum sales requirement (MSR): a subscription rate below which the issuance is withdrawn. Both features mitigate underpricing in our setting and, in special cases, can eliminate it altogether.

Our framework also yields new empirical predictions for best-effort equity offerings with an MSR. Each relies on the following fact: the optimal MSR is the subscription rate below which a share is worth less than its offer price. This choice of MSR maximizes an investor’s

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1 This quote appears in the SEC’s announcement of rule 15c2-4, which penalizes intermediaries who fail to distribute the proceeds of an offering to the issuer.

2 Kim and Weisbach [59] find that a dollar raised in an IPO leads to increases of 78¢ in R&D and 20¢ in capital expenditures over the succeeding four years. Firms that undergo successful IPOs also hire more workers in subsequent years (Babina, Ouimet, and Zarutskie [8]; Borisov, Ellul, and Sevilir [14]). IPOs raise substantial revenue, ranging from 29.3% to 52.5% of pre-IPO firm value (Spiess and Pettway [83, Table 2]; Brennan and Franks [16, Table 2]). Similarly, initial public debt offerings - a firm’s first public debt offering after an IPO - raised 19.7% of common equity in the sample of Datta, Iskandar-Datta, and Patel [27, Table II] and 42% in that of Burnie and Ogden [18, Table 1].

3 Some relevant studies are discussed in section 5.2.
incentive to subscribe by ensuring that she will be asked to buy a share if, and only if, doing so is \textit{ex-post} worthwhile.

We find, first, that this optimal MSR is decreasing in the fixed cost of subscribing (which may be due, e.g., to costs of due diligence). Intuitively, to entice investors to subscribe despite a higher fixed cost, the issuer must make the offered shares more valuable.\footnote{In our setting, she does so by offering a higher ownership stake per share.} This, in turn, lowers the subscription rate below which a share is worth less than its offer price: the optimal MSR falls.

We show also that the optimal MSR is increasing in the sensitivity of of the firm’s cash flow to its issuance revenue. To take a concrete example, suppose that - during the pre-IPO period - the issuer’s bank is told by regulators that it must raise its capital adequacy ratio. The bank restricts lending as a result. The issuer now must rely more on issuance revenue to fund investments. A given degree of undersubscription thus has a larger negative effect on the issuer’s share value. This raises the subscription rate below which a share is worth less than its offer price: the optimal MSR rises.\footnote{This holds fixed the ownership stake conferred by a single share. A more precise intuition, which does not rely on this assumption, appears after the formal result (section 6.3).}

Our theory helps resolve two empirical puzzles. The first is the finding that the inclusion of secondary shares in an IPO mitigates underpricing (Ang and Brau \cite{AngBrau2003}; Habib and Ljungqvist \cite{HabibLjungqvist2000}). In our setting, this is because secondary share revenues do not affect the issuer’s future cash flows. As the subscription rate has a smaller effect on share value, investors face less strategic risk; a smaller discount thus suffices to induce them to subscribe.\footnote{In practice, secondary shares make up a small portion of shares sold in IPOs: 14.5\% in Habib and Ljungqvist \cite{HabibLjungqvist2000, Table 1}, 16.4\% in Spiess and Pettway \cite{SpiessPettway2002, Table 2}, 29\% in Ang and Brau \cite{AngBrau2003, Table 1}, and 36.2\% in Huyghebaert and Van Hulle \cite{HuyghebaertVanHulle2010, Table 2}. This may be because their inclusion is interpreted as a negative signal (Ang and Brau \cite{AngBrau2003}).}

The second puzzle is Ritter’s \cite[Table 4, p. 273]{Ritter1991} finding that underpricing is much lower in firm-commitment IPOs than in best-effort IPOs: 14.8\% vs. 47.8\%.\footnote{Booth and Chua \cite[p. 299]{BoothChua1996} argue that best-effort offerings tend to be smaller and thus require more} Intuitively, in
a firm-commitment IPO, an underwriter commits to buy all unsold shares. By lowering the undersubscription risk faced by investors, this mitigates underpricing. However, it does not entirely eliminate it. Intuitively, in a firm-commitment issuance, the offer price is typically chosen on the eve of the offering, well after most investors have indicated their interest (Lowry, Michaely, and Volkova [63, pp. 219 ff.]). If interest is weak, the issuance may still go ahead with a lower share price and thus less valuable shares. An investor who expressed interest may feel pressure to subscribe in order to preserve access to future valuable issuances. Underpricing may be needed to entice investors to express interest despite this strategic risk.

Our finding that firms issue debt to minimize underpricing also gives a new foundation for Myers’ [67] pecking order hypothesis. The dominant foundation is based on asymmetric information; see, e.g., Biais and Mariotti [11], DeMarzo [31], DeMarzo and Duffie [32], DeMarzo, Frankel, and Jin [33], Leland and Pyle [62], Myers and Majluf [68], and Nachman and Noe [69]. However, these models also predict that any security sold in equilibrium will be fairly priced. They thus do not explain why, in practice, underpricing is larger for equity than for debt.

In our theory, underpricing results from the risk that undersubscription will deprive the issuer of needed capital. Relatedly, in Plantin [71], underpricing results from the risk that undersubscription will lead to an illiquid secondary market, while in Welch [88] the issuer underprices to encourage herding à la Scharfstein and Stein [80]. These studies also rely on a single selection criterion and do not consider security design.

In Allen and Faulhaber [3], Grinblatt and Hwang [45], and Welch [86], a privately informed issuer may underprice to signal optimism, thus obtaining more favorable terms in subsequent SEOs. Two other theories of underpricing (Rock [78] and Benveniste and Spindt...
are discussed below in our section on share rationing (section 6.2).

We study the problem of raising capital for a project whose returns are used to repay investors. Moreover, we rely on equilibrium selection criteria to predict investor behavior. Both ingredients are also present in Allen et al [2], Chakraborty, Gervais, and Yilmaz [21] (CGY), Goldstein and Pauzner [44] (GP), and Halac, Kremer, and Winter [48] (HKW). However, these prior papers each use a single selection criterion and do not study security design.\textsuperscript{10}

2 Base Model

We focus throughout on fixed-price, best-effort issuance mechanisms. Best-effort mechanisms and common in private placement offerings (Robbins [77, p. 2]) and were used in 35\% of the 1,028 IPOs in Ritter’s dataset [73, p. 270]. Dunbar [35, p. 63] finds that fewer than 8\% of best-effort IPOs have price revisions, which is consistent with our assumption of a fixed price.\textsuperscript{11}

There are two periods: $t = 1, 2$. The players consist of a single issuer and a set $I$ of \textit{ex-ante} identical agents. The agents may be either discrete ($I = \{1, ..., n\}$) or infinitesimal ($I = [0, 1]$). They collectively have some high but finite amount $\overline{p} > 0$ of wealth to invest, of which each agent has an equal share.\textsuperscript{12} All players are risk-neutral and fully rational; there is no discounting.

\textsuperscript{10}CGY use the PI criterion and restrict to portfolios of equity and warrants; HKW use the Unique Implementation criterion and study a setting (two cash flow realizations) in which debt and equity are equivalent; and Allen \textit{et al} and GP use the Laplace criterion and focus on demand deposit contracts. These criteria are defined below in section 3.

\textsuperscript{11}Moreover, in our setting the issuer knows \textit{ex ante} how much the agents are willing to pay so no price revision is needed to ensure full subscription.

\textsuperscript{12}The wealth constraint is assumed to be so high as to be nonbinding, except in section 6.2 where it may bind at the optimal scheme.
The issuer owns a random cash flow $Y \geq 0$ that is realized in period 2. She can raise the distribution of this cash flow by selling claims to it in period 1 in return for capital $K \in [0, \overline{p}]$. The cash flow $Y_\kappa$ that results from raising capital $K = \kappa$ is bounded and has the atomless distribution function\(^\text{13}\)

$$H (y|\kappa) \overset{d}{=} \Pr (Y_\kappa \leq y) \tag{1}$$

with associated survivor function

$$\overline{H} (y|\kappa) \overset{d}{=} 1 - H (y|\kappa) = \Pr (Y_\kappa > y). \tag{2}$$

Let

$$y_\kappa = \max \left\{ y \geq 0 : H (y|\kappa) = 0 \right\} \quad \text{and} \quad \overline{y}_\kappa = \min \left\{ y \geq 0 : H (y|\kappa) = 1 \right\} \tag{3}$$

denote the lower and upper bound, resp., on the support of $Y_\kappa$.\(^\text{14}\) As $H (\cdot|\kappa)$ is bounded and atomless,

$$0 \leq y_\kappa \leq \overline{y}_\kappa < \infty. \tag{4}$$

We assume higher capital $\kappa$ raises the cash flow distribution in the following sense.

**Hazard Rate Ordering (HRO)**

1. For any capital levels $\kappa' > \kappa$ both in $[0, \overline{p}]$, the ratio $\frac{\overline{H} (y|\kappa)}{\overline{H} (y|\kappa')}$ is decreasing in $y \in [y_\kappa, \overline{y}_\kappa]$.
2. The bounds $y_\kappa$ and $\overline{y}_\kappa$ on the conditional cash flow $Y_\kappa$ are continuous and nondecreasing in $\kappa \in [0, \overline{p}]$.
3. The function $\overline{H} (y|\kappa)$ is Lipschitz-continuous in $\kappa$, uniformly in $y$: there is a finite constant $\lambda_H$ such that for all $y \in [0, \overline{p}]$ and $\kappa, \kappa' \geq 0$, \[ |\overline{H} (y|\kappa') - \overline{H} (y|\kappa)| \leq \lambda_H (|\kappa| - |\kappa'|) \]

\(^\text{13}\)The notation “$\overset{d}{=}”$ denotes a definition. Capital $K$ is random since it depends on the agents’ subscription rate. The parameter $\kappa$ is a generic realization of $K$.

\(^\text{14}\)Since $Y_\kappa$ is bounded and atomless, $H (y|\kappa)$ is continuous in $y$ and satisfies $H (0|\kappa) = 0$ and, for high enough $y$, $H (y|\kappa) = 1$. Thus, the definitions in (3) are well-defined.
\[ \lambda_H \mid \kappa' - \kappa \].

Part 1 states that a rise in issuance revenue lowers the cash flow hazard rate: the probability that \( Y = y \) conditional on \( Y \geq y \).\(^{15}\) Part 2 states that a rise in issuance revenue does not lower the highest and lowest possible cash flows. Part 3 is a technical continuity property.

HRO is weaker than the monotone likelihood ratio property\(^{16}\) but stronger than another well-known property:

**First Order Stochastic Dominance (FOSD)** For any \( \kappa' > \kappa \) both in \([0, \bar{p}]\) and \( y \geq 0 \),

\[ \overline{H}(y|\kappa) \leq \overline{H}(y|\kappa') \] where the inequality is strict for \( y \in (y_\kappa, \overline{y}_{\kappa'}) \).

**Claim 1.** HRO implies FOSD.

In period 1, the issuer may stay out (offer no security), raising zero capital but retaining full rights to her cash flow: her payoff is then \( E[Y_0] \). Or she can offer a scheme \( s = (p, S) \) where \( p \in [0, \bar{p}] \) is the price per share and \( S \) is a monotone security.\(^{17}\)

**Definition 2.** A security \( S \) is *monotone* if both \( S(y) \) and \( y - S(y) \) are nonnegative and nondecreasing in \( y \).

\(^{15}\)More precisely, if the density \( \frac{\partial}{\partial y} \overline{H}(y|\kappa) \) exists, then part 1 of HRO implies that for \( \kappa' > \kappa \),

\[ 0 > \frac{\partial}{\partial y} \overline{H}(y|\kappa) = \overline{H}(y|\kappa') \left[ \frac{\partial}{\partial y} \overline{H}(y|\kappa') \overline{H}(y|\kappa) - \frac{\partial}{\partial y} \overline{H}(y|\kappa) \overline{H}(y|\kappa') \right]. \tag{5} \]

That is, the cash flow hazard rate \( \frac{\partial}{\partial y} \overline{H}(y|\kappa) / \overline{H}(y|\kappa) \) is decreasing in issuance revenue \( \kappa \). However, HRO is more general than (5) as it does not assume the density exists.

\(^{16}\)See Shaked and Shanthikumar [81, theorem 1.C.1].

\(^{17}\)A monotone security is one for which both the portion of the cash flow paid to investors and the portion retained by the firm are nonnegative and nondecreasing in the cash flow. Monotonicity is a common assumption in the security design literature; see, e.g., DeMarzo [31], DeMarzo and Duffie [32], DeMarzo, Frankel, and Jin [33], Frankel and Jin [40], Hart and Moore [50], and Nachman and Noe [69]. Examples of monotone securities include equity, standard debt, and warrants (call options). Monotonicity is justified by supposing the issuer has free disposal over her cash flow \( Y \) and can also contribute cash to inflate it. Hence, if her payout \( Y - S(Y) \) were decreasing in \( Y \), she would freely dispose of some cash in order to raise this payout. And if, alternatively, the payout \( S(Y) \) to the agents were falling in \( Y \), the issuer would contribute cash to inflate \( Y \), thus paying the agents less and raising her payout \( Y - S(Y) \) by more than the amount contributed.
Since a monotone security is nonnegative, the issuer never strictly prefers to offer a scheme with a zero price.\(^{18}\) We thus restrict attention to schemes \((p, S)\) with positive prices:

\[ p > 0. \] \(^{(6)}\)

On seeing the scheme \((p, S)\), the agents simultaneously decide whether or not to subscribe: to pay \(p\) for a share of security \(S\). Subscribing incurs a fixed cost\(^{19,20}\)

\[ c \geq 0, \] \(^{(7)}\)

which may capture, e.g., the time costs of due diligence. Let \(L \in [0, 1]\) denote the aggregate subscription rate: the proportion of agents who subscribe. If at least one agent subscribes but the fraction \(1 - L\) who do not is positive, the firm sells the remaining \(1 - L\) shares at a fixed discount \(\rho \in (0, 1)\).\(^{21,22}\) Accordingly, total capital raised in the issuance is

\[ K = Lp + (1 - L)(1 - \rho)p \in [0, p] \] \(^{(8)}\)

which equals the price \(p\) if all agents subscribe (if \(L = 1\)).

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\(^{18}\)Her payoff is \(E[Y_0 - S(Y_0)]\) from offering \((0, S)\) versus \(E[Y_0]\) from staying out. (The notation \(Y_\kappa\) is defined in \((1)\).) As \(S \geq 0\), the former payoff is never higher than the latter.

\(^{19}\)If the agents are discrete, each subscriber incurs a cost \(c/n\) and pays the issuer \(p/n\) for \(1/n\) shares, which entitle her to \(S(Y)/n\). If they are infinitesimal \((I = [0, 1])\), any set of agents of measure \(\varepsilon > 0\) that subscribes incurs the fixed cost \(ce\) and pays the issuer \(pe\), for which they receive \(\varepsilon\) shares which pay them \(S(Y)\varepsilon\) in aggregate. (Each agent decides independently whether or not to subscribe.)

\(^{20}\)We show in section 6.3 that a positive cost \(c\) prevents MSR schemes from eliminating underpricing. In the two other variants of our model, the fixed cost \(c\) plays no essential role.

\(^{21}\)Without the condition that at least one agent be willing to subscribe, an issuance would raise at least \((1 - \rho)p\) in revenue no matter how attractive it is. The issuer could thus raise infinite revenue by letting \(p\) go to infinity, which is absurd.

\(^{22}\)For instance, the unsold shares might be placed with a large investor who demands a discount based on a rule of thumb or who subsequently meddles in the firm’s decisions, lowering its cash flow à la Burkart, Gromb, and Panunzi [17]. Alternatively, the firm may retain the unsold shares in its treasury and plan to offer them later to a different set of investors, with the discount capturing uncertainty about the price and timing of this sale. Importantly, no shares are sold at a discount \(in equilibrium\) as the issuer will either stay out or offer a scheme \((p, S)\) that induces all agents to subscribe.
From the perspective of a given investor, let\(^{23}\)
\[
\ell \in \Lambda \overset{d}{=} \begin{cases} 
[0, 1] & \text{if agents are infinitesimal} \\
\lambda \overset{d}{=} \left\{ \frac{i}{n-1} : i = 0, \ldots, n-1 \right\} & \text{if agents are discrete}
\end{cases}
\] (9)
denote the proportion of the other investors who subscribe. We refer to \(\ell\) as the other-agent subscription rate. The relation between the aggregate and other-agent subscription rates is given by the function
\[
L = L(\ell) \overset{d}{=} \begin{cases} 
\ell & \text{if } I = [0, 1] \\
\frac{1+\ell(n-1)}{n} & \text{if } I = \{1, \ldots, n\}
\end{cases}
\] (10)
Using (8) and (10), we can express capital \(K\) as a function of \(\ell\):
\[
K = p r(\ell) \in [0, \bar{p}]
\] (11)
where we refer to
\[
r(\ell) \overset{d}{=} 1 - \rho + \rho L(\ell)
\] (12)
as the effective subscription rate: the number of shares that would have to be sold at the price \(p\) to yield issuance revenue \(K\). By (10), \(r(\cdot)\) is an increasing function from \([0, 1]\) to \([1 - \rho + \iota, 1]\) satisfying
\[
r(1) = 1
\] (13)
where
\[
\iota = \begin{cases} 
0 & \text{if } I = [0, 1] \\
\frac{1}{n} & \text{if } I = \{1, \ldots, n\}
\end{cases}
\] (14)
is the measure of a single agent. By (11), an agent’s expected payoff from subscribing to the

\(^{23}\)The notation “\(\overset{d}{=}\)” denotes a definition. In the definition of \(\lambda\), \(i\) is interpreted as the number of others who participate while \(n - 1\) is the total number of other investors.
scheme \( s = (p, S) \), given the other-agent subscription rate \( \ell \), is\(^{24}\)

\[
\pi_s(\ell) = E \left[ S \left( Y_{pr(\ell)} \right) \right] - \theta (p + c) \tag{15}
\]

where

\[
\theta > 0 \tag{16}
\]

is the gross risk-free interest rate. By (6), (7), and (16), the agents’ opportunity cost of subscribing is positive:

\[
\theta (p + c) > 0. \tag{17}
\]

We assume

\[
E [Y_0] > c\theta, \tag{18}
\]

so the agents are willing to subscribe if offered a 100% equity stake at a zero price.

The most capital the issuance can raise is \( p \). Thus, by Def. 2 and Claim 1, an investor’s payoff (15) from subscribing is bounded above by \( E [Y_p] - (p + c) \theta \). To avoid a trivial outcome, we restrict to prices \( p \) for which this upper bound is positive. This assumption, combined with (6), can be written as

\[
p \in \left( 0, \frac{E [Y_p]}{\theta} - c \right). \tag{19}
\]

3 Resolving Indeterminacy: Methodology

The monotonicity of \( S \) combined with FOSD implies that the payoff function defined in (15) is nondecreasing in the other-agent subscription rate \( \ell \). If it crosses zero from below, there are multiple equilibria: if all other agents are expected (not) to subscribe, it is a best response (not) to subscribe. For the issuer to choose an optimal scheme, she must therefore predict which equilibrium they will select for any possible scheme \( s \): she needs a theory of equilibrium selection.

\(^{24}\)By (1), \( Y_{pr(\ell)} \) is the random cash flow \( Y_\kappa \) that results from issuance revenue \( \kappa = pr(\ell) \).
To solve this problem, we use the new approach of Frankel [37]. He shows that under weak conditions given below, seven iconic equilibrium selection theories all lead to selection criteria of a simple common form:\textsuperscript{25}

\begin{equation}
\text{an agent (does not) subscribe if } \int_{\ell=0}^{1} \pi_s(\ell) \, d\Gamma(\ell) > (<) 0\tag{20}
\end{equation}

for some distribution $\Gamma$ that depends on the theory but not on $\pi_s$.\textsuperscript{26} While we will work with general fictional beliefs $\Gamma$ as well as the following two specific candidates for $\Gamma$:\textsuperscript{27}

\textbf{The Partial Implementation (PI) criterion} predicts that investors will play a best response to the belief $\Gamma^{\text{PI}}$ that all others will subscribe for sure: that $\ell = 1$ with probability one.

\textbf{The Laplace criterion} predicts that investors will play a best response to the belief $\Gamma^{\text{Laplace}}$ that all other-agent subscription rates are equally likely.

For the seven theories to imply criteria of the form (20), the payoff function $\pi$ must satisfy one of the following single crossing properties, which are due to Athey [7]. The first states that if the net payoff from subscribing is positive for one other-agent subscription rate $\ell$, then it is not negative for any higher rate:

\textbf{Weak SC1:} For all pairs $\ell' > \ell$, $\pi(\ell) > 0$ implies $\pi(\ell') \geq 0$.

The second, which is stronger, states that if the net payoff is nonnegative (positive) at one rate $\ell$, then it remains so at any higher rate:

\textbf{SC1:} For all pairs $\ell' > \ell$, $\pi(\ell) \geq (>0)$ implies $\pi(\ell') \geq (>0)$.

\textsuperscript{25}This section presents an abridged version of Frankel [37], explaining only the concepts and findings that we will need. Complete results, intuitions, and proofs appear in Frankel [37].

\textsuperscript{26}Rather than true beliefs, the fictional beliefs $\Gamma$ capture the weight that the theory assigns to different segments of the payoff function $\pi_s$ in making its predictions.

\textsuperscript{27}For the fictional beliefs associated with the other theories, see Frankel [37].
Claim 3. The following are sufficient conditions for the selection theories studied by Frankel [37] each to imply a criterion of the form (22).

1. Investors are discrete and the payoff function \( \pi \) satisfies weak SC1 on \( \lambda \).

2. Investors are infinitesimal and the payoff function \( \pi \) is Lipschitz-continuous on \([0,1]\) and satisfies SC1 on \((0,1)\). (This part excludes the theory of Kandori, Mailath, and Rob [58], which is defined only for discrete agents.)

Proof. See Table 1 in Frankel [37].

We study the base model and two extensions. In each variant, the issuer will select a scheme \( s \) from some compact set \( \Sigma \). For any scheme \( s \in \Sigma \), let

\[
\varphi_\Gamma (s) = \int_{t=0}^{1} \pi_s (\ell) d\Gamma (\ell)
\]

be an agent’s expected payoff from subscribing under the fictional beliefs \( \Gamma \). We will say that \( s \) is successful if \( \varphi_\Gamma (s) > 0 \): if the scheme induces the agents to subscribe under (20). We assume the issuer will either propose a successful scheme \( s \), getting some payoff \( U_s \), or stay out, getting \( E [Y_0] \).

However, an optimal successful scheme may not exist as the set \( O = \{ s \in \Sigma : \varphi_\Gamma (s) > 0 \} \) of successful schemes is not closed. Thus, we follow Frankel [38] in looking for a scheme with the following property.

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28 E.g., in the base model, \( \Sigma \) is the set of schemes \( (p, S) \) where \( p \in [0, \bar{p}] \) and \( S \) is monotone. We permit \( p = 0 \) for compactness; however, the issuer will never offer such a scheme as it involves giving the security away for free.

29 There are three cases by (20) and (21). If \( \varphi_\Gamma (s) \) is positive, all agents subscribe. If \( \varphi_\Gamma (s) \) is negative, no agents subscribe. As failed offerings are costly in practice (Dunbar [35]), we assume the issuer will not propose a scheme for which \( \varphi_\Gamma (s) < 0 \), preferring to stay out instead. If \( \varphi_\Gamma (s) \) is zero, the agents’ response is indeterminate. In two of the three versions of our model, the issuer can induce the agents to subscribe by sweetening the proposal infinitesimally (see Claims 13 and 27). We given an algorithm below to find such schemes, which are a good approximation to an optimal successful scheme. In the third version (minimum sales constraints), there are schemes \( s \) satisfying \( \varphi_\Gamma (s) = 0 \) that cannot be sweetened in this way. Frankel [38] shows that allowing an issuer to propose such schemes can lead to absurd results. Hence, we rule them out.
Definition 4. (Frankel [38]) A scheme \( s \in \Sigma \) is \textit{approximately optimal} if (a) there is no successful scheme \( s' \in \Sigma \) that satisfies \( U_{s'} > U_s \) and (b) for any \( \varepsilon > 0 \) there is a successful scheme \( s' \in \Sigma \) within \( \varepsilon \) of \( s \), such that \( |U_s - U_{s'}| < \varepsilon \).

An approximately optimal scheme \( s \) yields a tight upper bound \( U_s \) on the issuer’s payoff \( U_{s'} \) from any successful scheme \( s' \). Moreover, there are nearby successful schemes \( s' \) that give the principal payoffs \( U_{s'} \) near \( U_s \). Finally, an approximately optimal scheme always exists.\(^{30}\)

To find an asymptotically optimal scheme, we use the following procedure.

\textbf{Heuristic Search Procedure (HSP)} (Frankel [38]) (a) Specify an agent type (discrete or infinitesimal) and let \( \Lambda \) denote the set of feasible other-agent subscription rates \( \ell \). (b) Specify a nonempty set \( \Sigma \) of feasible schemes \( s \) and, for each scheme \( s \) in \( \Sigma \), a payoff function \( \pi_s : \Lambda \to \mathbb{R} \) for the agents and a payoff \( U_s \in \mathbb{R} \) that the issuer receives if all agents subscribe. (c) Specify a metric \( m \) on \( \Sigma \) and verify that the set \( \Sigma \) is compact and the maps \( s \to U_s \) and, for all \( \ell \in \Lambda \), \( s \to \pi_s (\ell) \) are continuous with respect to \( m \). (d) Show that the sufficient conditions in Claim 3 for the chosen agent type (discrete or infinitesimal) hold for any scheme \( s \in \Sigma \). (e) If the set \( O \) of successful schemes is empty, abort as the issuer cannot induce the agents to subscribe. (f) Let \( \Sigma' \) be the result of removing from \( \Sigma \) an arbitrary (and possibly empty) set of schemes that are not near any successful schemes.\(^{31}\) (g) Find a scheme \( s^* \) that maximizes \( U_s \) on \( \Sigma' \) subject to

\begin{equation}
\varphi_\Gamma (s) \geq 0.
\end{equation}

(h) Show that for every \( \delta > 0 \) there is a successful scheme \( s' \in \Sigma \) that is within \( \delta \) of \( s^* \).

\(^{30}\)One can show that a scheme is approximately optimal if and only if it maximizes \( U_s \) on the closure of \( O \). If, moreover, \( \Sigma \) is compact and \( U_s \) is continuous (which we will assume), the given maximum exists by the extreme value theorem. See Frankel [37] for details.

\(^{31}\)A scheme \( s \) is \textit{not near any successful schemes} if for some \( \delta > 0 \) there is no scheme \( s' \in O \) that is a distance less than \( \delta \) away from \( s \) under the metric \( m \).
Claim 5. (Frankel [38]) Assume steps (a)-(d) of HSP are satisfied. (A) If a scheme \( s^* \in \Sigma \) solves steps (e)-(h), it is approximately optimal in \( \Sigma \). (B) If \( s^* \) is approximately optimal in \( \Sigma \), then there is a way to delete schemes in step (f) such that \( s^* \) satisfies steps (e), (g), and (h).

We apply HSP to our base model and two extensions. In each case, we simplify by showing that step (g) implies step (h). For the MSR variant, schemes must be deleted in step (f) for HSP to have a solution.\(^{32}\)

The following result will imply that fair pricing is the best the issuer can do.

Claim 6. Assume steps (a)-(d) of HSP are satisfied. Let \( s^\dagger \in \Sigma \) satisfy \( \pi_{s^\dagger}(1) < 0 \). Then \( s^\dagger \) is not successful, is not the limit of any sequence of successful schemes, and is not asymptotically optimal.

4 Underpricing: An Intuition

An intuition for underpricing is as follows. If the issuer chooses to go ahead with an offering, she will offer her security at a price that the agents are willing to pay given their fictional beliefs.\(^{33}\) Hence the agents will all subscribe. On the other hand, all of the fictional beliefs except PI assign a positive probability to undersubscription. If the agents rely on such “pessimistic” fictional beliefs, the issuance must be underpriced to induce them to subscribe.\(^{34}\) While the price may appear to be too low \textit{ex post}, it is correct \textit{ex ante}: any higher price would lead the agents to choose the bad equilibrium in which no one subscribes.

This result is illustrated in Figure 1. An issuer offers a single unit of an equity security to a unit measure of agents.\(^{35}\) The security entitles an agent to a proportion \( \alpha \in [0, 1] \) of

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\(^{32}\)See section 6.3.

\(^{33}\)This relies on our assumption of symmetric information.

\(^{34}\)An exception can occur in the case of debt with a very low face value; see Theorem 18.

\(^{35}\)The equity assumption is for illustration only; the argument applies to any monotone security.
a future cash flow $Y$ whose expectation is rising in the amount of capital raised. We hold the security price $p$ fixed and let the issuer vary the equity share $\alpha$, which appears on the horizontal axis. The ray ABC gives the agents’ valuation of the security for each share $\alpha$ if all subscribe: if the firm raises capital $p$. The ray AFG gives their valuation if none subscribe: if the firm raises zero capital. Finally, the ray ADE gives their valuation when the subscription rate is distributed according to the fictional beliefs $\Gamma$, which we assume to lie between the first two cases.\footnote{Why do the agents’ valuations have the form of a ray? An agent’s valuation is simply the equity share $\alpha$ times the present value of the firm’s expected cash flow. This expectation, in turn, depends only on the distribution of the amount $pL$ of capital raised where $L \in [0, 1]$ denotes the proportion of agents who subscribe. As the distribution of $L$ is held fixed within each case, the expected cash flow does not vary with $\alpha$ (although it varies across cases). Hence, an agent’s valuation is a ray whose slope is the expected cash flow in the given case.}

![Figure 1: illustration of equilibrium underpricing result.](image)

An agent subscribes if doing so is optimal under her fictional beliefs: if her willingness...
to pay ADE is not less than the price \( p \). The issuer naturally chooses the lowest such equity share, denoted \( \alpha^* \). As this share induces all agents to subscribe, the security’s \textit{ex-post} market value is given by the height of point B. Since this height exceeds \( p \), the security is underpriced by the length of segment BD.

Now suppose the issuer tries to leave less money on the table by choosing an equity share slightly below \( \alpha^* \). The agents’ willingness to pay ADE is now below \( p \); they will not subscribe, so the \textit{ex-post} market value of the security is the height of point F. As this height is less than the price \( p \), the security is now \textit{overpriced} by the length of segment DF. In the model, overpricing is not observed in equilibrium as it follows a deviation. In practice, however, negative information may emerge during the first trading day. If this information is large enough, it may push a share’s fair value below its offer price: the issuance appears to have been overpriced. This fits the findings of Ritter and Welch [75, p. 1806], in which both negative and positive first-day returns occur while the latter are more prevalent.

5 Solving the Base Model

As noted above, we assume the issuer will either stay out, getting \( E[Y_0] \), or propose a successful scheme \( s = (p, S) \). Since such a scheme \( s \) induces all agents to subscribe \((L = 1)\), it raises capital \( K = p \) by (8). The issuer’s payoff from a successful scheme \( s \) is the expectation

\[
U_s = E[Y_p - S(Y_p)]
\]

(23)
of the resulting cash flow \( Y_p \) less the security payout \( S(Y_p) \). By (15), a subscriber’s payoff \( \pi_s(1) \) if all others subscribe \((\ell = 1)\) equals the expected security payout \( E[S(Y_p)] \) less the opportunity cost \( \theta(p+c) \) of subscribing. We will say that the scheme is \textit{fairly priced} if this

\footnote{This example assumes the agents subscribe if the issuer chooses \( \alpha = \alpha^* \) despite being indifferent under their fictional beliefs. As noted in section 3, our results do not rely on this assumption.}
payoff is zero: if
\[ p = V_s \equiv \frac{1}{\theta} E [S(Y_p)] - c. \] (24)
We refer to \( V_s \) as the *fair price* of \( s \) and to
\[ V_s - p \] (25)
as the *underpricing* of \( s \): the gap between the scheme’s fair price and its offer price \( p \).\(^{38} \) For a fixed capital target \( p \), maximizing the issuer’s payoff is equivalent to minimizing underpricing:

**Claim 7.** Let \( s = (p, S) \) and \( s' = (p, S') \) be two schemes with the same price \( p \). Then \( U_s > U_{s'} \) if and only if \( V_s - p < V_{s'} - p \). Moreover, if \( s \) and \( s' \) are both successful, then the issuer prefers the scheme that is less underpriced.

*Proof.* Immediate from (23) and (24). \( \Box \)

For any fictional beliefs \( \Gamma \), let the *fictional cash flow* \( Y_{p|\Gamma} \) be the random cash flow under the counterfactual belief that the other-agent subscription rate \( \ell \) has the distribution \( \Gamma \). It is defined by
\[ \Pr (Y_{p|\Gamma} \leq y) = G_{\Gamma} (y|p) \equiv \int_{\ell=0}^{1} \Pr (Y_{pr(\ell)} \leq y) d\Gamma (\ell) = \int_{\ell=0}^{1} H (y|pr(\ell)) d\Gamma (\ell) \] (26)
by (1). Intuitively, \( G_{\Gamma} (y|p) \) is the expectation, conditional on \( \ell \sim \Gamma \), of the distribution function of the cash flow \( Y_{pr(\ell)} \) that results when capital \( pr(\ell) \) is raised. By (26), the survival function of \( Y_{p|\Gamma} \) is given by
\[ \overline{G}_{\Gamma} (y|p) \equiv 1 - G_{\Gamma} (y|p) = \int_{\ell=0}^{1} \overline{H} (y|pr(\ell)) d\Gamma (\ell) \] (27)
where \( \overline{H} \) is defined in (2). More optimistic fictional beliefs weakly raise this function:

\[^{38} \] A fairly priced offering may still have a positive return on the issuance day. This occurs, e.g., if \( c > 0 \) and a share can be bought costlessly in the aftermarket (i.e., with no due diligence costs). In that case the offer price is given by (24) while the aftermarket price is \( \frac{1}{\theta} E [S(Y_p)] \): the price \( p \) rises by \( c \). However, defining the fair price as \( \frac{1}{\theta} E [S(Y_p)] \) seems less satisfactory as it is an unattainable goal as long as \( c > 0 \).
Claim 8. Fix a capital target \( p \in (0, \bar{p}] \). If \( \Gamma' \) first-order stochastically dominates \( \Gamma \), then \( Y^\Gamma_p \) first-order stochastically dominates \( Y^{\Gamma'}_p \): if \( \Gamma' (\ell) \leq \Gamma (\ell) \) for all \( \ell \in [0, 1] \), then \( \overline{G}_{\Gamma'} (y | p) \geq \overline{G}_\Gamma (y | p) \) for all \( y \).

Finally, the fictional cash flow \( Y^\Gamma_p \) that corresponds to PI beliefs \( \Gamma = \Gamma^{\text{PI}} \) is simply the cash flow that results if capital \( p \) is raised for sure:

\[
Y^{\Gamma^{\text{PI}}}_p = Y_p. \tag{28}
\]

We now write a security’s expected payout \( E \left[ S \left( Y^\Gamma_p \right) \right] \) under the fictional beliefs in terms of the survival function \( \overline{G}_\Gamma \) and show that more optimistic fictional beliefs \( \Gamma \) weakly raise this expected payout:

Claim 9. For any price \( p \) and monotone security \( S \),

1. the security’s expected security payout \( E \left[ S \left( Y^\Gamma_p \right) \right] \) under the fictional beliefs \( \Gamma \) can be written as \( \int_{\gamma=p}^{\gamma=0} \overline{G}_\Gamma (y | p) \, dS(y) \), which equals \( \int_{\gamma=p}^{\gamma=0} \overline{H} (y | p) \, dS(y) \) under PI beliefs \( \Gamma = \Gamma^{\text{PI}} \); and

2. if \( \Gamma' \) first-order stochastically dominates \( \Gamma \), then \( E \left[ S \left( Y^{\Gamma'}_p \right) \right] \) is not less than \( E \left[ S \left( Y^\Gamma_p \right) \right] \).

To find an optimal security, we now carry out the steps of HSP. For HSP(a), we allow both discrete or infinitesimal agents so \( \Lambda = [0, 1] \). For HSP(b), we fix a price \( p \in (0, \bar{p}] \) and let \( \Sigma \) denote the set \( \Sigma^p_0 = \{ p \} \times M \) of schemes with price \( p \) where

\[
M \overset{d}{=} \left\{ S : [0, \overline{y}_p] \rightarrow [0, \overline{y}_p] : S(y) \ & & S(y) \ are \ nonnegative \ & & \ are \ nondecreasing \right\} \tag{29}
\]

is the set of monotone securities. Further, let \( \pi_s \) and \( U_s \) be as given in (15) and (23). For HSP(c), we define the distance between any two schemes \( s = (p, S) \) and \( s' = (p, S') \) to be

\[
m(s, s') = \sup_{y \in [0, \overline{y}_p]} |S(y) - S'(y)|. \tag{30}
\]

We now verify the conditions of HSP(c):

\[39\]This is because PI beliefs \( \Gamma^{\text{PI}} \) put all of their weight on \( \ell = 1 \) and \( pr(1) \) equals \( p \) by (10) and (12).
Claim 10. The set $\Sigma^p_0$ is compact and the functions $U_s$ and, for all $\ell \in \Lambda$, $s \to \pi_s(\ell)$ are continuous with respect to $m$.

Turning to HSP(d), the conditions of Claim 3 hold for both agent types:

Claim 11. For any scheme $s$ in $\Sigma^p_0$, $\pi_s$ is Lipschitz-continuous and satisfies SC1 on $[0, 1]$.

For HSP(e), let $O$ be the set $O^p_0 \subseteq \Sigma^p_0$ of schemes $s$ for which $\varphi_{\Gamma}(s) > 0$. By (15) and (26), for any scheme $s = (p, S) \in \Sigma^p_0$ we can rewrite (21) as

$$\varphi_{\Gamma}(s) = E[S(Y^p_{\Gamma})] - \theta(p + c).$$

(31)

By monotonicity, $S(y) \leq y$. Thus, by (31),

Claim 12. $O^p_0$ is nonempty if and only if the agents will all subscribe to a 100% equity stake ($S(y) \equiv y$) for the price $p$. This holds if and only if

$$p < \frac{1}{\theta}E[S(Y^p_{\Gamma})] - c.$$  

(32)

To satisfy HSP(e), we assume (32).

We remove no schemes in HSP(f). Next we show that HSP(g) implies HSP(h).

Claim 13. If the scheme $s = (p, S)$ satisfies (22) then for any $\delta > 0$ there is a successful scheme $s'$ within $\delta$ of $s$.

To find an approximately optimal scheme, it thus suffices to solve HSP(g). We can rewrite the constraint (22) in HSP(g) as

$$p \leq \frac{1}{\theta}E[S(Y^p_{\Gamma})] - c$$

(33)

by (31): the price $p$ does not exceed the security’s value under the fictional beliefs. Since, moreover, the issuer prefers less underpriced schemes, HSP(g) is equivalent to minimizing underpricing subject to (33):

Claim 14. A scheme $s$ satisfies HSP(g) if and only if there is no alternative scheme $s'$ that satisfies (33) and has lower underpricing (25).
Furthermore, the best one can hope for is fair pricing:\textsuperscript{40}

**Claim 15.** For any scheme $s$ that satisfies (33), underpricing (25) is nonnegative.

Finally, the constraint (33) must bind at any approximately optimal scheme:

**Claim 16.** For any scheme $s = (p, S)$ that solves $HSP(g)$, the constraint (33) binds:

$$ p = \frac{1}{\theta} E [S(\bar{Y}_p)] - c. $$

(34)

Intuitively, if (33) holds strictly at $(p, S)$ then the issuer can replace $S$ with $(1 - \varepsilon)S$. For small enough $\varepsilon > 0$, (33) will still hold but the issuer’s payoff $U_s$ will be higher.

### 5.1 The PI Case

We will say that a scheme $s = (p, S)$ is \textit{fairly priced} if underpricing $V_s - p$ is zero. By (24), this means that the expected payout of the security under full subscription equals the agents’ opportunity cost of subscribing:

$$ E[S(\bar{Y}_p)] = \theta (p + c). $$

(35)

Fair pricing suffices for approximate optimality in the PI case:\textsuperscript{41}

**Claim 17.** Assume $\Gamma = \Gamma^{PI}$. Then $s = (p, S)$ is approximately optimal if and only if it is fairly priced.

Intuitively, agents who rely on the PI criterion play a best response to the correct belief that all others will subscribe. Thus, they will buy any fairly priced security. And, by Claim 7, the issuer is indifferent among all fairly priced securities as well.

\textsuperscript{40}Intuitively, (33) states that the price $p$ does not exceed the security’s value under the fictional beliefs which, in turn, does not exceed the scheme’s fair price by (24) and part 2 of Claim 9.

\textsuperscript{41}The proof relies on the fact that, in light of our prior results, approximate optimality is equivalent to $HSP(g)$ by Claim 5.
5.2 The Non-PI Case

We now turn to the non-PI case: $\Gamma \neq \Gamma^{\text{PI}}$. We show that standard debt is approximately optimal and thus minimizes underpricing among schemes that satisfy (33).\textsuperscript{42} Intuitively, debt minimizes the investors’ strategic uncertainty as its payout is constant except in the rare case of default. Hence, it is the least underpriced security. We show, further, that a debt issuance can be fairly priced if it is not too large. These predictions are consistent with the literature: stock IPOs are underpriced (Ritter and Welch [75]), while the average IPDO is more or less fairly priced (Datta, Iskandar-Datta, and Patel [27]). Moreover, debt is indeed more prevalent than equity: nonconvertible debt made up 82% of all new capital raised worldwide from 1990 to 2001 (Henderson, Jegadeesh, and Weisbach [53, Table 2, p. 69]).\textsuperscript{43,44} DeAngelo and Roll [30, p. 405]; Denis and McKeon [34]; Im, Mayer, and Sussman [55, Table 4]).

Let $y_{\Gamma}^p$ denote the lower bound of the cash flow support under the fictional beliefs $\Gamma$.\textsuperscript{45} The prevalence of debt supports not only our model, but also prior explanations such as adverse selection (section 1) and costly state verification (Townsend [84]). Further research is needed to determine the relative importance of these various theories.

We will say that a debt security $S(y) = \min\{y, D\}$ is risk-free if it will surely be repaid if all subscribe: if $D \leq y_{\Gamma}^p$. Otherwise we will say that it is risky. With this terminology, we can now state the main result of this section:\textsuperscript{46}

**Theorem 18.** Fix a price $p \in (0, \overline{p}]$ that satisfies (32). Let

$$S(y) = \min\{D, y\} \quad (36)$$

\textsuperscript{42}By Claim 14, solving HSP(g) is equivalent to minimizing underpricing subject to (33).

\textsuperscript{43}Debt also rises during investment spikes, suggesting that firms prefer to use debt to fund new investments (Bargeron, Denis, and Lehn [9]; DeAngelo, DeAngelo, and Whited [29, pp. 255-7]; DeAngelo and Roll [30, p. 405]; Denis and McKeon [34]; Im, Mayer, and Sussman [55, Table 4]).

\textsuperscript{44}The prevalence of debt supports not only our model, but also prior explanations such as adverse selection (section 1) and costly state verification (Townsend [84]). Further research is needed to determine the relative importance of these various theories.

\textsuperscript{45}It equals the lower bound $y_{\text{pr}}(\ell_{\Gamma})$ of the cash flow when revenue $p \times \ell_{\Gamma}$ is raised, where $\ell_{\Gamma}$ is the lower bound of the support of $\Gamma$. Under the five non-Pareto beliefs $\Gamma$ surveyed in section 3, $\ell_{\Gamma}$ equals zero so $y_{\Gamma}^p$ equals $y_{\ell}^0$.

\textsuperscript{46}The comment in n. 41 applies also to Theorem 18.
be the standard debt contract with face value $D$ given implicitly by

$$p = \frac{1}{\theta} E \left[ \min \{ D, Y_p^\Gamma \} \right] - c. \quad (37)$$

The scheme $(p, S)$ is approximately optimal and minimizes underpricing. Moreover:

1. If $D \leq y_p^\Gamma$, the security $S$ is risk-free and fairly priced.

2. If $D \in \left( y_p^\Gamma, y_p \right]$, the security $S$ is risk-free and underpriced.

3. If $D > y_p$, the security $S$ is risky and underpriced.

Intuitively, in case 1 the face value $D$ lies below the lowest cash flow $y_p$ that can occur under the true beliefs $\Gamma^{\text{PI}}$: the security is risk-free. Moreover, the face value also lies below the lowest cash flow $y_p^\Gamma$ that can occur under the fictional beliefs $\Gamma$. Hence, the agents’ selection criterion also treats the security as risk-free: the agents fairly price the security. In case 2, the security is still risk-free since $D < y_p$. However, as $D$ now exceeds $y_p^\Gamma$, the agents’ fictional beliefs place positive weight on default, leading them to value the security as if it were risky and thus to undervalue it. Finally, in case 3, $D$ exceeds $y_p$ so the security is risky. Moreover, while default is possible under both $\Gamma$ and $\Gamma^{\text{PI}}$ (as $D > y_p \geq y_p^\Gamma$), it is more likely under $\Gamma$: the security is undervalued as in case 2.

We conclude with two empirical predictions regarding the yield spread $\frac{D_p}{p} - \theta$ in new bond issuances. We show that this spread is increasing in due diligence costs, as proxied by the time cost $c$, but decreasing in the availability of alternative funding sources - which we proxy by a rise in the sensitivity of the future cash flow to revenue shortfalls in the issuance.

More precisely, let $H'(y|\kappa)$ be an alternative distribution that satisfies HRO and let $\overline{H'} = 1 - H'$ be the corresponding survival function. We focus on distributions $H'$ that are more sensitive to the cash flow in the following sense. First, the hazard ratio is weakly more sensitive to issuance revenue under $H'$ than $H$:

$$\overline{H'}(y|\kappa') \overline{H}(y|\kappa) \leq \overline{H}(y|\kappa') \overline{H'}(y|\kappa) \text{ for any } \kappa' > \kappa \text{ and } y \geq 0, \quad (38)$$

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which is equivalent to $\frac{H(y|\kappa)}{H'(y|\kappa')} \leq \frac{H(y|\kappa)}{H'(y|\kappa')}$ when both ratios are well-defined. Second, the shift from $H$ to $H'$ is not “good news” (in a FOSD sense) for the cash flow conditional on full subscription:

$$H'(y|p) \leq H(y|p) \text{ for all } y \geq 0. \quad (39)$$

**Claim 19.** Fix the capital target $p$. Let $c' \geq c$ and let $H$ and $H'$ be two distributions that satisfy HRO, (39), and (38). Then the yield spread $\frac{D}{p} - \theta$ given by the face value $D$ in (37) is weakly higher under $(c', H')$ than under $(c, H)$, and is strictly so if $c' > c$.

The empirical literature indeed suggests that bond yields in new issuances are rising in the cost of due diligence (Andres, Betzer, and Limbach [5]; Datta, Iskandar-Datta, and Patel [28]; Fridson and Garman [41]). However, we can find no test of the effect of a higher need for issuance revenue.

## 6 How to Sell Equity if You Must

Many firms do sell equity, despite the fact (both empirically and in our model) that equity leads to more underpricing.\(^{47}\) Reasons may include diversification (Zingales [89]; Chemmanur and Fulghieri [23]), inducing information production (Chemmanur [22]), exploiting stock overvaluation (Lucas and McDonald [64]; Ritter [74]), creating public shares to pay for acquisitions (Brau and Fawcett [15]), and preserving borrowing capacity (DeAngelo, DeAngelo, and Whited [29]). For issuers to leave so much money on the table, they must have a strong incentive to issue equity that is unrelated to issuance revenue. Thus, we now study two common devices that can mitigate underpricing in equity issuances. As equity is fairly priced in the PI case (Claim 17), we restrict to the non-Pareto case.

In the first device, the issuer can raise the number of shares an investor can request so that

\(^{47}\text{See Datta, Iskandar-Datta, and Patel [27] and Ritter and Welch [75] for empirical support.}\)
shares must be rationed once the subscription rate surpasses a given threshold.\footnote{In our base model, (a) there is a unit number of shares and investors and (b) each investor demands either zero or one shares: oversubscription cannot occur. In the extension, we relax (b): an investor demands either zero or \( z > 1 \) shares, so oversubscription occurs if more than \( 1/z \) investors subscribe.} Once this occurs, a further rise in the subscription rate entails a fall in each subscriber’s allotment such that issuance revenue is unchanged. This insurance effect mitigates underpricing. However, there is an offsetting effect: when the subscription rate is low enough to avoid rationing, a subscriber receives her full share request but - as the issuance raises less capital in this event - the shares are not worth as much. While this effect worsens underpricing, it is smaller than the first effect for all but one of the selection criteria we consider.\footnote{Even for the criterion that is an exception, share rationing mitigates underpricing if the fixed subscription cost is small enough; see section 6.2.} This may help explain the empirical prevalence of ex-post share rationing.\footnote{See Amihud, Hauser, and Kirsh [4, p. 146] and Cornelli and Goldreich [26, p. 1419].}

In the second device, the issuer specifies a minimum sales requirement (MSR): a minimum subscription rate below which the issuance is withdrawn. By ensuring that subscribers are not forced to buy shares when issuance revenue is low, such a scheme can mitigate underpricing. In the extreme case of a zero time cost \( (c = 0) \), the issuer can eliminate underpricing by setting the MSR to 100\%.\footnote{This assumes that the upper bound of the support of the agents’ fictional beliefs is one, which holds for all of the criteria of section 3 except Unique Implementation.} With a positive time cost \( c \), in contrast, a rise in the MSR has a second effect: it leads an investor to defray her fixed, positive time cost over a shrinking set of subscription rates at which the issuance goes through. Because of this effect, underpricing remains under an MSR. Moreover, an MSR below 100\% is typically optimal as the time cost effect grows as the MSR rises. This is consistent with the empirical literature: most MSRs are positive but below 100\% (Cho [24, Table 3]; Welch [87, Figure 1]).

For the second device, we also show two comparative statics results: if the issuer’s cash flow is less sensitive to issuance revenue and/or the agents’ time cost is higher, a lower MSR
is optimal. These predictions find indirect support in two findings of Welch [87]. The first is
that issuers with higher sales revenue tend to choose lower MSRs (Welch [87, Table 5]). As
Welch notes, an issuer with higher sales revenue depends less on issuance revenue to fund its
projects. In our setting, this means that the cash flow is less sensitive to issuance revenue,
so the insurance benefit of a higher MSR is smaller. This leads the issuer to choose a lower
MSR.

The second finding is a positive association between the issuer’s MSR and the compensa-
tion paid to the underwriter (Welch [87, Table 6]). This can be reconciled with our model
via the omitted variable of underwriter reputation. Intuitively, reputable underwriters both
charge high fees and certify the issuer, lowering an investor’s time costs (which include due
diligence).\(^{52}\) And as noted, a lower due time cost leads the issuer to raise her MSR. Our
model thus predicts that underwriter compensation and MSRs will be positively related.

### 6.1 Common Assumptions

In the base model, the issuer proposes a price \(p\) and a monotone security \(S\). The “base
model with equity” will refer to the restriction of this model to equity securities:

\[
S(y) = \alpha y \quad \text{for some constant } \alpha \in (0, 1).
\] (40)

In this restricted version, the agents’ participation constraint in step (g) of HSP can be
written as

\[
p \leq \frac{\alpha}{\theta} E\left[\frac{Y^{\Gamma}}{p}\right] - c.
\] (41)

We strengthen (19) by restricting to positive prices \(p\) for which a 100% equity stake (\(\alpha = 1\))
satisfies the constraint (41) strictly:

\[
p \in \left(0, \frac{E\left[\frac{Y^{\Gamma}}{p}\right]}{\theta} - c\right).
\] (42)

\(^{52}\)Evidence for certification comes from Lee and Masulis [61], who find that underwriter reputation is
negatively associated with earnings management by the issuer.
Under PI fictional beliefs, equity can be sold at a fair price in the base model (Claim 17). We thus rule out that case: we assume throughout this section that
\[ \Gamma \neq \Gamma^{PI}. \] (43)

It follows that, unlike debt, equity is always underpriced in the base model:\textsuperscript{53}

**Claim 20.** Assume \( p > 0 \). 1. The true expected cash flow \( E[Y_p] \) exceeds the cash flow \( E[Y^\Gamma_p] \) that is expected under the fictional beliefs. 2. Any equity security \( S \) that solves HSP\((g)\) is underpriced.

Let
\[ \Upsilon^p_\ell = E[Y_{pr(\ell)}], \] (44)
donote the expected cash flow conditional on the price \( p \) and the other-agent subscription rate \( \ell \). By (13), the issuer raises \( pr(1) = p \) if all others subscribe so
\[ \Upsilon^p_1 = E[Y_p]. \] (45)

Moreover, a higher subscription rate raises \( \Upsilon^p_\ell \) and thus the fair value of equity:

**Claim 21.** For any price \( p > 0 \), the conditional expected cash flow \( \Upsilon^p_\ell \) is positive and is Lipschitz-continuous and increasing in \( \ell \).

We modify the base model with equity in two ways. In both, if all investors subscribe, each pays \( p \) for one share of the security - just as in the base model with equity. Reasoning as at the start of section 2 and using (40) and (45), each extension thus has the following properties.

**Claim 22.**

\textsuperscript{53}Intuitively, the increment \( dS(y) = \alpha dy \) in the payout of equity is positive at all realizations \( y \) as \( \alpha > 0 \). For \( y \) above the lower bound \( y^\Gamma_p \) on the cash flow given \( \Gamma \), the agents require a discount to purchase such increments since (by FOSD) their fictional beliefs \( \Gamma \) place a lower probability on cash flows above such \( y \) than do the correct beliefs \( \Gamma^{PI} \). This argument can fail for debt \( S(y) = \min\{D, y\} \) as the increment \( dS(y) \) is zero for cash flow realizations \( y \) above the face value \( D \). In particular, if \( D < y^\Gamma_p \), the security can be sold at a fair price since, under \( \Gamma \), the cash flow will always exceed the face value.

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1. The issuer’s realized payoff $U_s$ from proposing a successful scheme $s$ is

$$U_s = (1 - \alpha) \Upsilon_{1}^p.$$  

(46)

2. A subscriber’s payoff if all others subscribe is

$$\pi_s(1) = \alpha \Upsilon_{1}^p - \theta (p + c).$$  

(47)

3. The scheme $s$ is underpriced (resp., fairly priced) if

$$0 < (=) V_s - p = \frac{\pi_s(1)}{\theta}$$  

(48)

where $V_s = \frac{\alpha \Upsilon_{1}^p}{\theta} - c$ is the fair price.

As $U_s$ and $V_s$ depend only on $p$ and $\alpha$, and since $U_s$ (resp., $V_s$) is increasing (decreasing) in $\alpha$, an optimal scheme is one that minimizes underpricing:

**Claim 23.** In the base model with equity and each extension thereof, a scheme $s$ maximizes the issuer’s payoff $U_s$ over a set if and only if it minimizes underpricing $V_s - p$ over the same set.

Let the pure benefit per share function

$$\psi^p_\alpha (\ell) = \alpha \Upsilon_{\ell}^p - p\theta$$  

(49)

denote the expected value of a share in period 2 less the monetary cost $p\theta$ of this share. It is thus the “pure benefit” of obtaining a single share, ignoring any time cost.\(^{54}\)

By (47) and (49), the payoff if all subscribe can be written as

$$\pi_s(1) = \psi^p_\alpha (1) - c\theta.$$  

By Claim 6, we can restrict to schemes $s$ that satisfy $\pi_s(1) \geq 0$ or, equivalently,

$$\psi^p_\alpha (1) \geq c\theta,$$  

(50)

\(^{54}\)The payoff function $\pi_s(\ell)$ is simply $\psi^p_\alpha (\ell)$ less the time cost $c\theta$ in the base model with equity. The relation between the two functions is more complex in the two devices that we will study.
which by (49) pins down the set of admissible equity stakes:

\[ \alpha \in [\underline{\alpha}, 1] \]  

(51)

where

\[ \alpha = \frac{(p + c) \theta}{\Upsilon_1^p} \]  

(52)

is the unique solution to

\[ \psi_\alpha^p (1) = c\theta \]

by (49). As \( \alpha \) lies in \((0, 1)\) by (42), the set (51) has positive measure. With these assumptions, we have:

**Claim 24.** Fix \( p \) satisfying (42).

1. For any \( \ell \), \( \psi_\alpha^p (\ell) \) is continuous and increasing in \( \alpha \), and ranges from \(-p\theta < 0\) at \( \alpha = 0 \) to \( \Upsilon_\ell^p - p\theta \) at \( \alpha = 1 \).

2. For any \( \alpha > 0 \), \( \psi_\alpha^p (\ell) \) is continuous and increasing in \( \ell \in [0, 1] \).

**Proof.** Part 1 holds by (49) and Claim 21; part 2 holds by Claim 21.

We now turn to the two extensions. In each, a 100% equity stake for the price \( p \) will be feasible and will induce subscription by (42). Thus, HSP(e) is satisfied in each extension.

**6.2 First Device: Ex-Post Share Rationing**

Many IPOs are oversubscribed, leading to share rationing (n. 50). Underwriters seem to view rationing as desirable:

“Discussions with investment bankers indicate that they perceive that an offer should be two to three times oversubscribed to create a ‘good IPO’.” (Lowry, Michaely, and Volkova [63, p. 223])
What is the advantage of rationing? Our model suggests an answer: the prospect of *ex-post* rationing reduces *ex-ante* strategic uncertainty among the investors. Intuitively, once the subscription rate is high enough that shares must be rationed, a further rise in this rate has no effect on the amount of capital raised by the issuance. This can mitigate underpricing, helping the issuer. This theory supplements two answers from the prior literature:\(^{55}\)

- In Rock’s [78] winner’s-curse theory, the anticipation that high-quality issuances will be rationed lowers the willingness to pay of uninformed traders. In order to elicit the participation of these traders, issuers must underprice. This theory has empirical support\(^{56}\) but raises the question of why issuers do not take steps to mitigate the winner’s curse - e.g., by capping share requests *ex ante*.\(^{57}\)

- In Benveniste and Spindt [10] and Sherman [82], issuers use underpricing to induce investors to reveal private information about the firm, rewarding those who do so with larger share allocations. While this theory has empirical support,\(^{58}\) there is also evidence that underwriters allocate underpriced shares to their own profitable clients and mutual funds.\(^{59}\) Moreover, there is an alternative mechanism - auctions - that

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\(^{55}\)We focus here on models of share rationing. While these overlap with models of underpricing, they do not coincide. For instance, the security in our base model is underpriced but not rationed. Other underpricing theories are discussed in section 1.

\(^{56}\)See Aggarwal, Prabhala, and Puri [1], Amihud, Hauser, and Kirsh [4], and Michaely and Shaw [66].

\(^{57}\)Brennan and Franks [16] suggest that issuers use *ex-post* rationing to prevent the formation of large external blockholders and thus to avoid monitoring. Indeed, dispersed ownership may also strengthen incentives for managerial initiative (Burkart, Gromb, and Panunzi [17]) and lead to greater liquidity in the secondary market (Plantin [71]). However, *ex-ante* caps would also give these benefits while avoiding the winner’s curse. Moreover, external blockholders can act as a useful check on managers (Bolton and von Thadden [12]), so a founder might welcome them as a way to commit to good management.

\(^{58}\)For instance, Hanley [49] finds that offer price rises are followed by greater underpricing. Cornelli and Goldreich [25, 26] find that bidders who include limit prices receive more shares and that these limit prices affect the issue price.

\(^{59}\)Goldstein, Irvine, and Puckett [43], Jenkinson, Jones, and Suntheim [57], and Reuter [72] find that issuers sell underpriced shares to their profitable clients, while Ritter and Zhang [76] find that they place the shares with their own mutual funds.
aggregates information efficiently without the need for underpricing (Pesendorfer and Swinkels [70]).

In the base model, each subscriber requests a single share; since the numbers of shares and agents are equal, rationing cannot occur. To permit rationing, we now let each investor ask for \( z \geq 1 \) shares, where the cap \( z \) is announced by the issuer prior to the issuance. If more than \( 1/z \) investors subscribe, demand exceeds supply: shares must be rationed. Since issuance proceeds in this range are fixed, strategic uncertainty is reduced.

In addition to a price \( p > 0 \) and equity stake \( \alpha \), the issuer now chooses a number \( z \in [1, \bar{p}/p] \) of shares to offer to each agent.\(^61\) We focus on the case of infinitesimal agents: \( L = \ell \).\(^62\) If total share demand \( \ell z \) does not exceed the unit supply of shares, each subscriber gets \( z \) shares. Else shares are rationed: each subscriber gets \( \frac{1}{\ell} \) shares, so the total number \( \ell \times \frac{1}{\ell} \) of shares sold is one.

Let

\[
q = \begin{cases} \ell z & \text{if } \ell \leq 1/z \\ 1 & \text{if } \ell \geq 1/z \end{cases}
\]  \hspace{1cm} (53)

denote the aggregate number of shares that the subscribers get. As in the base model, the remaining shares are sold at a discount \( \rho \): revenue \( K \) equals \( p r(q) \) where \( r(q) \) is defined in (12). Thus by (44), the expected cash flow conditional on \( q \) is \( \Upsilon_q^p \).

If we evaluate the pure benefit per share (49) at the aggregate number \( q \) of shares sold,

---

\(^60\)This point is made also by Ritter and Welch [75, n. 10 (p. 1810)] and Lowry, Michaely, and Volkova [63, p. 241].

\(^61\)The upper bound on \( z \) is from the agents’ wealth constraint: a subscriber’s wealth \( \bar{p} \) must suffice to pay \( p \) per share for \( z \) shares.

\(^62\)An equivalent way to model rationing is to reduce the number of shares below one, while continuing to let each investor request one share; details available on request.

\(^63\)The assumption merely simplifies notation; the results hold qualitatively in the discrete case as well.
to subscribers in (53) and multiply the result by the number of shares

\[
\frac{q}{\ell} = \begin{cases} 
    z & \text{if } \ell \leq 1/z \\
    \frac{1}{\ell} & \text{if } \ell \geq 1/z 
\end{cases}
\]  \tag{54}

that a subscriber is allotted, we obtain the \textit{pure benefit} of subscribing:\footnote{The notation “a \wedge b” means \(\min\{a, b\}\).}

\[
\phi^p_\alpha (\ell | z) = \left( z \wedge \frac{1}{\ell} \right) \psi^p_\alpha (\ell z \wedge 1) = \begin{cases} 
    z \psi^p_\alpha (\ell z) & \text{if } \ell \leq 1/z; \\
    \frac{1}{\ell} \psi^p_\alpha (1) & \text{if } \ell \geq 1/z.
\end{cases}
\]  \tag{55}

The payoff \(\pi_s (\ell)\) from subscribing is simply the pure benefit (49) of subscribing, minus the time cost \(c\theta\):

\[
\pi_s (\ell) = \phi^p_\alpha (\ell | z) - c\theta. \tag{56}
\]

For HSP(a), the agents are infinitesimal so \(\Lambda = [0, 1]\). For HSP(b), we assume a fixed price \(p\) satisfying (42) and restrict to equity stakes \(\alpha\) satisfying (51). So that a scheme respects the investors’ wealth constraint, we restrict to caps \(z \in [1, \bar{p}/p]\).\footnote{To agree to buy up to \(z\) shares, an investor’s wealth \(\bar{p}\) must be at least \(pz\).} Let \(\Sigma\) denote the set of all such schemes for the fixed price \(p\). Finally, let the payoff functions \(U_s\) and \(\pi_s\) be given by (46) and (56), respectively.

For HSP(c), let the distance \(m(s, s')\) between \(s\) and any scheme \(s' = (p, \alpha', z')\) be the maximum difference between the parameters, \(\max\{|\alpha - \alpha'|, |z - z'|\}\). HSP(c) then holds:

\textbf{Claim 25.} \textit{The set \(\Sigma\) is compact and the functions \(U_s\) and, for all } \(\ell \in \Lambda\), \(s \to \pi_s (\ell)\) \textit{are continuous with respect to } \(m\).

For HSP(d), we show:

\textbf{Claim 26.} \textit{For } \(s \in \Sigma\), \(\pi_s\) \textit{is Lipschitz-continuous on } \([0, 1]\) \textit{and satisfies SC1 on } \((0, 1)\).

For HSP(e), by (42) a 100\% equity scheme with cap \(z = 1\) is in \(O\).\footnote{This is the scheme \(s = (p, 1, 1)\).} We remove no schemes in HSP(f). We next show that HSP(g) implies HSP(h).
Claim 27. Let $s$ solve $HSP(g)$. For any $\delta > 0$ there is a successful scheme $s'$ within $\delta$ of $s$.

In light of the above results and Claim 5, to find an approximately optimal scheme it suffices to solve $HSP(g)$. Moreover, by Claim 6 and part 3 of Claim 22, no overpriced scheme can solve $HSP(g)$. Thus, we look instead for a fairly priced scheme.

In a fairly priced scheme, the equity share $\alpha$ is pinned down by the capital target $p$: by (48), it must be

$$\alpha = \frac{(p + c) \theta}{\Upsilon_1^p},$$

which is independent of the cap $z$. It remains to find a cap $z$ that induces the investors to subscribe for the given pair $(p, \alpha)$.

Figure 2: Ex-post rationing scheme that eliminates underpricing under Laplacian fictional beliefs $\Gamma_{Laplace}^\text{Laplace}$.

Figure 2 illustrates the effect of rationing. Two pure benefit curves $\phi^p_\alpha(\ell|z)$ are depicted: one for the base case in which a subscriber asks for a single share ($z = 1$) and one for an ex-
post rationing scheme in which a subscriber requests three shares \((z = 3)\). By \((56)\), the net payoff \(\pi_s (\ell)\) is given by the vertical gap between the pure benefit curve and the horizontal dashed line at a height of \(c\theta\). Thus, to satisfy the participation constraint \((22)\), the integral \(\int_{\ell=0}^{1} \phi^p_{\alpha} (\ell|z) \, d\Gamma (\ell)\) of the pure benefit curve must be at least \(c\theta\).

This condition is violated in the base case. For instance, under Laplacian beliefs the integral of the pure benefit curve is then \(I-B-F\) which is clearly less than the integral \(A+E+H+I\) under the \(c\theta\) line. For other non-PI beliefs, the argument is as follows.\(^{67}\) We have chosen the equity share \(\alpha\) so the issuance would be fairly priced were the investors to subscribe: the pure benefit curve meets the \(c\theta\) line at \(\ell = 1\). But since the pure benefit curve in the base case is increasing in \(\ell\), it must lie below \(c\theta\) at all rates \(\ell < 1\). And non-PI beliefs must put positive weight on some rates \(\ell < 1\). So \((22)\) fails: the scheme is not successful.

How does this change with a cap \(z = 3\)? For rates \(\ell \in \left[0, \frac{1}{3}\right]\), each subscriber now gets three shares: the vertical scale of the pure benefit curve is multiplied by three while the horizontal scale is divided by three. Under Laplacian beliefs, the integral of the new curve over \([0, \frac{1}{3}]\) equals the integral of the old curve over \([0, 1]\): in the figure, \(D+E-B-C\) equals \(I-B-F\). For general fictional beliefs, it is ambiguous.

On the other hand, rationing also creates a new area \(\ell \in (\frac{1}{3}, 1)\) that unambiguously favors subscription. In this area, the issuance is fully funded while each subscriber gets more than one share, raising her pure benefit above \(c\theta\).

We identify two cases in which there exists a fairly priced rationing scheme that satisfies \((22)\) and thus induces the agents to subscribe. The first is Laplacian fictional beliefs. In this case, as noted, the area under the upwards sloping portion of the pure benefit curve is constant. And we show that the area \(G+H+I\) under the downwards sloping portion is proportional to \(\ln z\) which goes to infinity as \(z\) does. Thus, if the agents’ wealth \(\bar{p}\) is high enough there exists a cap \(z\) for which the area under the whole pure benefit curve exceeds

\(^{67}\) We rule out PI beliefs in \((43)\).
The agents subscribe. The second case is when the fictional beliefs rule out other-agent participation rates in a neighborhood $[0, \ell')$ of zero. In this case, for any cap $z > 1/\ell'$, the integral in (22) puts weight only on rates $\ell > \frac{1}{z}$, for which the pure benefit exceeds $c\theta$: the agents subscribe.\textsuperscript{68}

The formal result is as follows, in which we weaken the Laplace assumption slightly.\textsuperscript{69}

**Proposition 28.** Assume one of the following conditions:

1. there are constants $0 < c_1 < c_2 < \infty$ such that 
   \[
   \frac{\Gamma(\ell') - \Gamma(\ell)}{\ell' - \ell} \in (c_1, c_2) \text{ for all } 0 \leq \ell < \ell' \leq 1; \text{ or } \tag{58}
   \]
2. there is an $\ell' \in (0, 1)$ such that $\Gamma(\ell') = 0$.

Then for any time cost $c > 0$ and any capital target $p > 0$ that satisfies (32), there exists a $p^* < \infty$ such that if the agents’ wealth $p$ exceeds $p^*$, an ex-post rationing scheme $(p, \alpha, z)$ exists that is approximately optimal and fairly priced.

While the result does not hold for all fictional beliefs, it does for Laplacian beliefs which have some experimental support (Heinemann, Nagel, and Ockenfels [51, 52]). This may help explain the empirical prevalence of oversubscribed IPOs.

### 6.3 Second Device: a Minimum Sales Requirement (MSR)

In the vast majority of best-effort equity offerings, there is a minimum share requirement (MSR): a subscription rate below which the issuance is canceled (Cho [24, Table 3]; Welch [87, Fig. 1]). We now extend the base model with equity to include this feature.

\textsuperscript{68}While we know of no non-PI fictional beliefs with this property, the those associated with the theory of Kandori, Mailath, and Rob [58]) converge to a point mass on $\ell = 1/2$ in the limit as the number of agents goes to infinity (Frankel [37]). For such limiting fictional beliefs, any cap $z > 2$ eliminates underpricing.

\textsuperscript{69}Among the fictional beliefs in section 3 only Laplacian beliefs satisfy (58). However, in the future economists may identify selection criteria whose beliefs $\Gamma$, while not Laplacian, satisfy the condition as well.
If the time cost \( c \) is zero and the upper bound of the support of the fictional beliefs \( \Gamma \) is one,\(^7^0\) then an MSR scheme exists that eliminates underpricing. One simply sets the MSR to one, thus eliminating any strategic uncertainty, and reduce the equity stake \( \alpha \) to the point at which the offering is fairly priced. This does not work with a positive time cost \( c \), as this cost must now be defrayed over the set of subscription rates at which the issuance goes through. In this case, an MSR below one is generally optimal and the scheme does not eliminate underpricing. We also derive two comparative statics results: the MSR is decreasing in the time cost \( c \) and rising in the sensitivity of the expected cash flow \( E [Y_\kappa] \) to issuance revenue \( \kappa \). The findings of Welch \cite{87} support these two predictions.

As the payoff function \( \pi_s \) in an MSR scheme can be discontinuous, we assume agents are discrete (see Claim 3). In addition to a price \( p \) and an equity share \( \alpha \), the issuer now chooses an aggregate subscription rate \( L_0 \in \{ \frac{i}{n} : i = 1, ..., n \} \) below which the issuance is withdrawn.\(^7^1\) By (10), this corresponds to an other-agent subscription rate of \( \ell_0 = \frac{L_0 n - 1}{n - 1} \in \lambda \). Relying on this one-to-one relation, we assume that the issuer chooses \( \ell_0 \) rather than \( L_0 \).

If the issuance goes through, a subscriber incurs the time cost \( c \theta \) in return for the pure benefit \( \psi^p_\alpha (\ell) \) of buying a share. If it is canceled, she just incurs the time cost. Hence, the investors’ payoff function is

\[
\pi_s (\ell) = \begin{cases} 
-c\theta & \text{if } \ell < \ell_0; \\
\psi^p_\alpha (\ell) - c\theta & \text{if } \ell \geq \ell_0.
\end{cases}
\]  

(59)

We now step through HSP. For HSP(a), the agents are discrete so \( \Lambda = \lambda \). For HSP(b), we assume a fixed price \( p \) satisfying (42) and let \( \Sigma \) denote the set of all schemes \( s = (p, \alpha, \ell_0) \) for equity stakes \( \alpha \) satisfying (51) and MSRs \( \ell_0 \) in \( \lambda \). Let the payoff functions \( U_s \) and \( \pi_s \) be given by (46) and (59), respectively. For HSP(c), let the distance \( m (s, s') \) be-

---

\(^7^0\)This is so for five of the six criteria studied in Frankel \cite{37}.

\(^7^1\)If \( L_0 = 0 \), the issuance is never withdrawn, which is equivalent to the base model with equity. We thus exclude this case.
tween $s$ and any scheme $s' = (p, \alpha', \ell_0)$ be the maximum distance between parameters, 
\[ \max \{|\alpha - \alpha'|, |\ell_0 - \ell'_0|\}. \]
We have:

**Claim 29.** The set $\Sigma$ is compact and the functions $U_s$ and, for all $\ell \in \Lambda$, $s \to \pi_s(\ell)$ are continuous with respect to $m$.

For HSP(d) we show:

**Claim 30.** For any scheme $s$ in $\Sigma$, the function $\pi_s(\ell)$ satisfies weak SC1.

For HSP(e), $O$ is nonempty as it contains the scheme $(p, 1, 0)$ by (42) and Claim 12. A difference comes in HSP(f) where - unlike in the other two variants of the model - we must delete schemes for HSP(g) to imply HSP(h). Let
\[ \ell_\Gamma = \min \{ \ell' \in \lambda : \Gamma(\ell') = 1 \} \] (60)
be the highest other-agent subscription rate that can occur under the fictional beliefs $\Gamma$. Since the agents are discrete, $\Gamma$ puts positive weight on $\ell_\Gamma$. In HSP(f), we eliminate any scheme $s = (p, \alpha, \ell_0)$ that does not satisfy
\[ \ell_0 \leq \ell_\Gamma. \] (61)

**Claim 31.** A scheme that violates (61) is not near any successful scheme.

To see why we must remove schemes that violate (61), suppose the time cost $c$ is zero and let $s$ equal $(p, 0, \ell_0)$ for some $\ell_0 \in \lambda$ that exceeds $\ell_\Gamma$. The scheme asks the agents to give the issuer $p$ for nothing. It thus maximizes the issuer’s full-subscription payoff $U_s$ over $s \in \Sigma$. And since $c = 0$, the scheme satisfies (22) since, under the fictional beliefs, a subscriber buys zero shares. However, it is absurd that investors would agree to pay $p > 0$ for nothing. Moreover, if not dropped in HSP(f), the scheme $s$ would be selected in HSP(g) but fail HSP(h): the procedure would not produce a solution.

As a result of the deletions in HSP(f), HSP(g) implies HSP(h):

**Claim 32.** Let $s$ solve HSP(g). For any $\delta > 0$ there is a successful scheme $s'$ within $\delta$ of $s$. 36
Substituting (59) into the constraint (22) and using (21) yields the inequality
\[ c\theta \leq \int_{\ell_0}^{1} \psi_{\alpha}^p (\ell) dT (\ell) \tag{62} \]
where \( \psi_{\alpha}^p (\ell) \) is the pure benefit per share function defined in (49). By (46), to solve HSP(g) we must minimize the equity stake \( \alpha \). Since, moreover, the integral is increasing in \( \alpha \), for each \( \alpha \) we must choose the MSR \( \ell_0 \leq \ell \in \lambda \) that maximizes the integral. First define
\[ x_\alpha = \min \{ x \in [0, 1] : \psi_{\alpha}^p (x) \geq 0 \}. \tag{63} \]
Since, by (49) and Claim 21, \( \psi_{\alpha}^p (x) \) is increasing in \( x \), this is the cutoff for \( x \) below which \( \psi_{\alpha}^p (x) \) is negative. Key properties of this cutoff are given by the following result.

**Claim 33.** Fix \( p > 0 \). For each equity share \( \alpha \in [0, 1] \), \( x_\alpha \) is the unique element of \([0, 1]\) for which \( \psi_{\alpha}^p (\ell) \geq (\leq) 0 \) if and only if \( \ell \geq (\leq) x_\alpha \). The function \( x_\alpha \) is continuous in \( \alpha \). There are four cases.\(^{72}\)

1. If \( \psi_{\alpha}^p (0) > 0 \) then \( x_\alpha \) is zero, is locally constant in \( \alpha \), and satisfies \( \psi_{\alpha}^p (x_\alpha) > 0 \).

2. If \( \psi_{\alpha}^p (0) = 0 \) then \( x_\alpha \) equals zero and a fall (rise) in \( \alpha \) leads it to rise (resp., not to change). It satisfies \( \psi_{\alpha}^p (x_\alpha) = 0 \).

3. If \( \psi_{\alpha}^p (0) < 0 < \psi_{\alpha}^p (1) \), then \( x_\alpha \) lies in \((0, 1)\), is decreasing in \( \alpha \), and satisfies \( \psi_{\alpha}^p (x_\alpha) = 0 \).

4. If \( \psi_{\alpha}^p (1) = 0 \) then \( x_\alpha \) equals one and a rise (fall) in \( \alpha \) leads it to fall (resp., not to change). It satisfies \( \psi_{\alpha}^p (x_\alpha) = 0 \).

**Proof.** By (49) and (51), \( \psi_{\alpha}^p (1) \geq c\theta \geq 0 \) so \( x_\alpha \) exists. Moreover, \( \psi_{\alpha}^p (x) \) is continuous and increasing in \( x \) by Claim 24, so \( \psi_{\alpha}^p (x) \geq (\leq) 0 \) if and only if \( x \geq (\leq) x_\alpha \). The rest is a straightforward consequence of (63) and Claim 24. \( \square \)

\(^{72}\)The case \( \psi_{\alpha}^p (1) < 0 \) is ruled out by (50).
Let
\[
\lambda(x) = \min \{ \ell \in \lambda : \ell \geq x \}
\]
denote the result of rounding \(x\) up to the closest element of \(\lambda\). As \(\lambda(x)\) is nondecreasing in \(x\), \(\lambda(x_\alpha)\) is nonincreasing in \(\alpha\) by Claim 33. Since, by (61), we cannot choose an MSR over \(\ell_\Gamma\), the optimal feasible MSR is
\[
\ell_0^\alpha = \min \{ \lambda(x_\alpha), \ell_\Gamma \}.
\]
Intuitively, this MSR ensures that a subscriber will pay for a share only when doing so raises her payoff: only when her pure benefit per share is positive. By Claim 33,
\[
c\theta \leq \int_{\ell = \ell_0^\alpha}^1 \psi_\alpha^p(\ell) d\Gamma(\ell).
\]
This reduces HSP(g) to a one-dimensional problem: we seek the equity stake \(\alpha\) for which the constraint (66) binds.\(^{73}\)

---

**Figure 3**: Effect of a Minimum Subscription Rate (MSR). Number \(n\) of investors is assumed large enough that \(\lambda\) can be approximated as the unit interval \([0,1]\). Time cost \(c\) is zero in panels 1 and 2 and positive in panel 3. Upwards sloping dashed line in each panel is pure benefit function \(\psi_\alpha^p(\ell)\), which coincides with payoff function \(\pi(\ell) = \psi_\alpha^p(\ell) - c\theta\) in panels 1 and 2 (in absence of MSR) as \(c = 0\).

Some examples appear in Figure 3. In each panel, the upwards sloping dashed lines are pure benefit functions \(\psi_\alpha^p\). Panel 1 assumes a time cost of \(c = 0\). The payoff function \(\pi\) with no MSR is BDE. With an MSR of \(\ell_0\), the payoff function is ACDF, which satisfies weak

\(^{73}\)Such an \(\alpha\) must exist as the scheme \((p,1,0)\) satisfies (66) strictly by (42) and Claim 12.
SC1 but not SC1 as it drops from zero to negative at $\ell_0$. The optimal MSR $\ell_0^*$ is where the pure benefit per share function BDEF crosses the horizontal axis.\(^{74}\) While the resulting payoff function AEF satisfies strategic complementarities, to show that $\ell_0^*$ is optimal one must determine the investors’ choice in response to payoff functions such as ACDF that do not.

Our first result is that there is an MSR scheme that eliminates underpricing if the time cost $c$ is zero and the upper bound of the support of the fictional beliefs is one:

$$\ell_\Gamma = 1.$$  \hspace{1cm} (67)

This is illustrated in Panel 2. The MSR is $\ell_0 = 1$: unless all subscribe, the issuance is withdrawn. The equity stake is chosen to ensure fair pricing:

$$\alpha = \frac{p\theta}{\upsilon_1^\theta}. \hspace{1cm} (68)$$

The pure benefit curve $\psi_\alpha^p (\ell) = \alpha \upsilon_1^p - p\theta$ thus hits zero at $\ell = 1$: it is the dashed line CF. So the integral in (66) is zero which equals $c\theta$: the constraint (66) is satisfied. By (59), the full-subscription payoff $\pi_s (1)$ equals the gap $\psi_\alpha^p (1) - c\theta$ between the pure benefit per share and the time cost. This gap is zero so the issuance is fairly priced by (48).\(^{75}\)

Our second result is that an MSR cannot eliminate underpricing if the time cost $c$ is positive. This is illustrated in Panel 3, which assumes Laplacian beliefs. The integral in (66) is simply the area $A_2 + A_3$ under the ACD curve. For the constraint (66) to hold, this must not be less than $c\theta = A_1 + A_2$; that is, we must have $A_3 \geq A_1$. Since $A_1 > 0$, this forces point D to lie strictly above point E. But the gap between D and E equals $\psi_\alpha^p (1) - c\theta = \pi_s (1)$ which is proportional to underpricing $V_s - p$ by part 3 of Claim 22. Thus, an MSR scheme cannot eliminate underpricing when $c > 0$.

\(^{74}\)We assume the upper bound $\ell_\Gamma$ of the support of $\Gamma$ lies somewhere to the right of this intersection.

\(^{75}\)The role of (67) is to ensure that there exists a nearby successful scheme, as required by HSP(h). To produce such a scheme, raise the equity stake slightly to $\alpha'$ and lower the MSR accordingly to $\ell_0^\alpha$. The integral in (66) is now the integral of the ADE curve, which is positive by (67): the original scheme $(p, \alpha, \ell_0^\alpha)$ is asymptotically optimal.
Our results are as follows. By (44), $\Upsilon_{p, i}^{p}$ is the highest expected cash flow that investors can expect under their fictional beliefs.

**Proposition 34.** Fix $p$ satisfying (42).

1. If $c = 0$, the scheme $s = (p, \alpha_0, \ell_1)$ is approximately optimal where $\alpha_0 = p\theta / \Upsilon_{p, i}^{p}$. It yields underpricing
   
   $$V_s - p = p \left[ \Upsilon_{1}^{p} / \Upsilon_{i}^{p} - 1 \right]$$
   
   which is zero (resp., positive) if $\ell_1 = (\ell_1^{\ell})$.

2. If $c > 0$, the scheme $s_c = (p, \alpha_c, \ell_0^{\alpha_c})$ is approximately optimal where $\alpha_c$ is the unique equity stake $\alpha \in [\alpha, 1]$ that satisfies $c\theta = \int_{\ell = \ell_0}^{\ell_1} \psi_{\alpha}^{p} (\ell) \, d\Gamma (\ell)$. Underpricing is $\alpha_c \Upsilon_{1}^{p} / \theta - p + c$ which is positive and falls continuously to zero as $c$ does. The equity stake $\alpha_c$ is increasing in $c$ while the MSR $\ell_0^{\alpha_c}$ is weakly decreasing in $c$.

3. Let the alternative cash flow $\tilde{Y}$ be more sensitive to issuance revenue:

   $$E \left[ \tilde{Y}_{\kappa'} \right] - E \left[ \tilde{Y}_{\kappa} \right] > E \left[ Y_{\kappa'} \right] - E \left[ Y_{\kappa} \right] \text{ for each pair } \kappa' > \kappa \geq 0.$$  
   
   If, for fixed $c > 0$, $(p, \alpha_c, \ell_0^{\alpha_c})$ and $(p, \alpha_c, \ell_0^{\alpha_c})$ are approximately optimal schemes for $\tilde{Y}$ and $Y$, respectively, then $\ell_0^{\alpha_c} \geq \ell_0^{\alpha_c}$.

The proposition provides two comparative statics results. First, part 2 shows that the optimal MSR is weakly decreasing in the time cost $c$. Intuitively, to compensate an investor for a higher time cost, a higher equity stake $\alpha$ must be offered. This expands the range of other-agent subscription rates at which the pure benefit per share curve is positive. As the optimal MSR is the lowest such rate, it falls. We can also carry out a thought experiment using Panel 3 of Figure 3 to see this effect. A rise in $c$ raises the horizontal $c\theta$ dashed line. This causes $A_1$ to grow and $A_3$ to shrink. To restore the equality $A_1 = A_3$, the pure benefit per share line $BCD$ must shift upwards: the equity stake $\alpha$ must rise. This moves the intersection $C$ to the left: the MSR falls.
Second, the optimal MSR is weakly increasing in the sensitivity of the cash flow to issuance revenue (Proposition 34, part 3). Intuitively, this is because a rise in this sensitivity raises strategic risk which makes a higher MSR optimal. We can again see this using Panel 3 of Figure 3. Suppose the expected cash flow $E[Y_\kappa]$ becomes more sensitive to issuance revenue $\kappa$. If the MSR does not change, the BCD line rotates clockwise while point C stays fixed. This shrinks area $A_1$ and expands area $A_3$. To restore equality between these two areas, the BCD line must then shift to the right, which entails a rise in the MSR.

These predictions find support in the empirical literature. Welch [87, Table 5] finds that in best-effort offerings, firms with higher sales revenue tend to choose lower MSRs. His explanation is that such firms do not rely as much on issuance revenue to fund their projects. This is consistent with our finding that the MSR is decreasing in the the sensitivity of the issuer’s cash flow to issuance revenue.

Welch [87, Table 6] also finds that the MSR chosen in practice is positively related to underwriter fees. This may also be explained by our results. Intuitively, a higher quality underwriter both certifies issuer quality better, leading to a lower time cost $c$ of subscription, and charges a higher fee. By Proposition 34, the lower time cost leads in turn to a higher optimal MSR. This predicts a positive association between underwriter fees and the MSR.

7 Discussion

Firms raise substantial capital from selling primary securities. This capital can be invested to raise the firms’ future cash flows which, in turn, leads to higher distributions on the same securities. Thus, investors in a public offering face strategic uncertainty. In this setting, which securities are best for the issuer? And if equity must be sold, how best to structure the offering?

We study these questions in a stylized setting. An issuer owns a stochastic cash flow. She designs and sells a monotone security whose distributions are funded by the cash flow.
If the issuance raises more capital, the distribution of the cash flow is higher, which makes the securities more valuable. This leads to strategic complementarities in the decision to subscribe. In order to select among multiple equilibria, investors can use any criterion from a large set.

We show that investors who face strategic uncertainty require a discount in order to be willing to subscribe.\textsuperscript{76} Standard debt is optimal as it minimizes this underpricing. Intuitively, as long as an issuer of debt does not default, the security payout does not depend on her cash flow. Debt thus minimizes the negative impact of strategic uncertainty on the agents’ valuation of the security. Indeed, empirical studies have found that public debt offerings are less underpriced than stock IPOs which, in turn, are less underpriced when they contain more secondary shares (whose proceeds accrue to insiders rather than to the firm). We show, moreover, that the optimal yield spread on new bond issuances is increasing in due diligence costs but decreasing in the availability of alternative funding sources.

We also study two common devices in equity offerings. The first is \textit{ex-post} share rationing: if too many investors subscribe, the issuance is deemed “oversubscribed” and each subscriber’s share allotment is reduced proportionally. We show that rationing eliminates underpricing in IPOs for some equilibrium selection criteria, if investors are sufficiently wealthy.

The second device is the minimum sales requirement (MSR): if the subscription rate falls below some threshold, the issuance is canceled. If the time cost of subscribing is zero then, under a mild additional condition on the investors’ equilibrium selection criterion, underpricing can be eliminated by setting the MSR to 100%. If these conditions fail, an MSR can mitigate but not eliminate underpricing. Moreover, the optimal MSR is decreasing in investors’ time costs of subscribing and in the availability of alternative funding sources.

\textsuperscript{76}Under limited conditions, a security may be fairly priced (see Theorem 18).
References


